**Mathematics, Computing and Technology**
**M303 Further Pure Mathematics**

Specimen Solutions

# Formative solutions for TMA02 202xJ

## TMA 02

**Solution to Question 8** – 9 marks

(a) (i) We proceed by contradiction.
Assume that $n$ is prime. Then $\sigma(n) = \sigma(p) = 1 + p$ by
***Proposition 1.7, HB pg 42***.
Therefore $4n + \sigma(n) = 4p + 1 + p = 5p + 1$.
Therefore if $4n + \sigma(n)$ is divisible by 5 and $n$ is a prime $p$ we have that $5p + 1$ is divisible by 5 and this is impossible. Therefore $n$ cannot be prime.

(ii) Now let $n = p^2$. By ***Proposition 1.7, HB pg 42*** we have that $\sigma(p^2) = 1 + p + p^2$. Therefore $4n + \sigma(n) = 4p^2 + 1 + p + p^2 = 5p^2 + 1 + p$. This means that if $4n + \sigma(n)$ is divisible by 5 then we must have $5p^2 + 1 + p$ divisible by 5 and so $1 + p$ is divisible by 5. In other words, $p$ is equal to 4 (mod 5).

(iii) We know that $n$ cannot be prime and that $n$ is 4 mod 5 if it is a prime squared. So we can check non primes, $n = 4, 6, 8, 9, 10, 12$ and 14, and find that $\sigma(14) = 24$ and $4 \times 14 + 24 = 80$; or we can check numbers that are equal to a prime squared where that prime is 4 mod 5. The first prime that is also 4 mod 5 is 19. Calculating gives $4 \times 19^2 + \sigma(19^2) = 4 \times 19^2 + 1 + 19 + 19^2 = 1825$. So possible answers are 14 or 19. There will be others.

(b) Let $p$ and $q$ be distinct primes.
Then $pq + \sigma(pq) = pq + 1 + p + q + pq = 2pq + 1 + p + q$
***Proposition 1.7, HB pg 42***.
So if $pq + \sigma(pq)$ is even then we must have $1 + p + q$ even and so $p + q$ must be odd. Therefore the pairs are $(2, q)$ for any odd prime $q$.

(c) Let $m = p_1^{m_1} \cdots p_r^{m_r}$. Then $\tau(m) = (m_1 - +1) \cdots (m_r + 1)$
***Proposition 2.3 of Chapter 2, HB pg 17***. Therefore if $\tau(m)|m$ we must have $(m_1 + 1) \cdots (m_r + 1)|m$. But since $m$ is odd, we must have $m_i - 1$ being odd for all $i$. Therefore $m_i$ is even, say $m_i = 2k_i$. Then
$m = p_1^{m_1} \cdots p_r^{m_r} = p_1^{2k_1} \cdots p_r^{2k_r} = (p_1^{k_1} \cdots p_r^{k_r})^2$ as required.

**Solution to Question 9** – 8 marks

We need to check the group axioms, G1,G2,G3 and G4
***Definition 1.1 of Chapter 5, HB page 24***.

**Closed**

If $f$ and $g$ are both continuous on $[0, 1]$ then so is $(f + g)$ so it is closed (this is a standard result from, for example, the sum rule from M208).

**Identity**

Let $f_0$ be defined by $f_0(x) = 0$ for all $x \in [0, 1]$. Then for any $g \in C[0, 1]$ we have $(g + f_0)(x) = g(x) + 0 = g(x)$ and $(f_0 + g)(x) = 0 + g(x) = g(x)$ and so $f_0$ is the identity.

**Inverses**

Let $-f$ be defined by $(-f)(x) = -f(x)$ for all $x \in [0, 1]$. Then for any $f \in C[0, 1]$ we have $(f - f)(x) = f(x) - f(x) = 0 = f_0(x)$ and
$(-f + f)(x) = -f(x) + f(x) = 0 = f_0(x)$ and so $-f$ is the inverse of $f$.

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**Associative**

$((f + g) + h)(x) = (f + g)(x) + h(x) = (f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$
(by associativity in $\mathbb{R}$) $= (f + (g + h))(x)$.

Moreover, we have

$$sr^2 = r^{11}sr = r^{22}s = r^{10}s$$

and, similarly, $sr^{2k} = r^{12-2k}s$ for $k = 2, 3, 4, 5$, and so

$$sH = \{s, r^2s, r^4s, r^6s, r^8s, r^{10}s\} = Hs\,.$$

The remaining six elements of $D_{12}$ must form the fourth coset (by the proof of Lagrange's Theorem and
***Proposition 3.7(b) of Chapter 5, HB page 27***), and so

$$rsH = \{rs, r^3s, r^5s, r^7s, r^9s, r^{11}s\} = Hrs\,.$$

Therefore $H$ is a normal subgroup of $D_{12}$ by
***Proposition 3.7(b) of Chapter 5, HB page 27***.

(c) (i) The cosets are

i. $H$
ii. $rH = \{r, r^3, r^5, r^7, r^9, r^{11}\}$
iii. $sH = \{s, r^2s, r^4s, r^6s, r^8s, r^{10}s\}$
iv. $rsH = \{rs, r^3s, r^5s, r^7s, r^9s, r^{11}s\}$

The Cayley table is as follows.

| | $H$ | $rH$ | $sH$ | $rsH$ |
|---|---|---|---|---|
| $H$ | $H$ | $rH$ | $sH$ | $rsH$ |
| $rH$ | $rH$ | $H$ | $rsH$ | $sH$ |
| $sH$ | $sH$ | $rsH$ | $H$ | $rH$ |
| $rsH$ | $rsH$ | $sH$ | $rH$ | $H$ |

(ii) $D_{12}/H$ is not cyclic, since there is no element of order 4, or all non-identity elements have order 2.

***Chapter 7, Exercise 1.4 and solution, pages 156 and 194–5***

***Chapter 5, Exercise 3.10 and solution, pages 37 and 74***

**Solution to Question 11** – 10 marks

(a) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_6$ and $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$
***Corollary 2.12 of Chapter 6, HB page 32***.

(b)

| Group | Order of group | Isomorphic groups | Orders of elements | Cyclic? Abelian? |
|---|---|---|---|---|
| $D_{12}$ | 24 | None | 1, 2, 3, 4, 6, 12 | Non abelian |
| $\mathbb{Z}_3 \times \mathbb{Z}_4$ | 12 | $\mathbb{Z}_{12}$ | 1, 2, 3, 4, 6, 12 | Cyclic |
| $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ | 12 | $\mathbb{Z}_2 \times \mathbb{Z}_6$ | 1,2,3,6 | Abelian, non cyclic |

***Chapter 6, Exercise 2.8 and solution, pages 111 and 142***

***Chapter 6, Exercise 2.9 and solution, pages 112 and 142***

***Chapter 6, page 112, Theorem 2.11***

(c) $\mathbb{Z}_3 \times \mathbb{Z}_4$ and $D_{12}$.

**Solution to Question 12** – 8 marks

(a) (i) Here $K$ is not a union of conjugacy classes of $D_{10}$, hence by ***Proposition 3.12(a) of Chapter 5, HB page 28*** it is not normal in $D_{10}$. Then $D_{10}$ is not isomorphic to the direct product $H \times K$ (that is, it is not the internal direct product of $H$ and $K$) by ***Theorem 1.5 of Chapter 6, HB page 30***.

(ii) Since both $H$ and $K$ are unions of conjugacy classes, they are normal subgroups of $D_{10}$ by ***Proposition 3.12(b) of Chapter 5, HB page 28***. Moreover, their intersection is $\{e\}$, and $HK$ contains the generators $r$ (because $r = r^6 r^5$) and $s$, and so $HK = D_{10}$. Hence $D_{10} \cong H \times K$ by ***Theorem 1.5 of Chapter 6, HB page 30***.

***Chapter 6, Exercise 1.7 and solution, pages 96 and 138***

(b) Let $p$ and $q$ be distinct primes.

(i) Since $p^2$ and $q$ are coprime, $\mathbb{Z}_{p^2q}$ is isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ by ***Corollary 2.12 of Chapter 6, HB page 32***.

(ii) By ***Proposition 2.8 of Chapter 6, HB page 32***, the highest order of an element of $\mathbb{Z}_{p^2q}$ is $p^2q$. The maximum order of an element of $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$ is $pq$. Hence the two groups are not isomorphic by ***Theorem 4.6 of Chapter 5, HB page 27***.

**Alternatively:**

$\mathbb{Z}_p \times \mathbb{Z}_p$ is not cyclic by ***Theorem 2.11 of Chapter 6, HB page 32***, hence $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$ has a non-cyclic subgroup, and it cannot be cyclic by ***Theorem 2.3 of Chapter 6,HB page 31***. Since $\mathbb{Z}_{p^2q}$ is cyclic, we have

$$\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q \ncong \mathbb{Z}_{p^2q}\,.$$

***Chapter 5, page 44, discussion following Exercise 4.9***

***Chapter 6, Exercise 2.9 and solution, pages 112 and 142***